

# Properties of factorial cumulant to factorial moment ratio \*

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## Abstract

It is shown that the ratio of factorial cumulant moments to factorial moments for a multiplicity distribution truncated in the tail reveals oscillations in sign similar to those observed in experimental data. It is suggested that this effect be taken into account in the analysis of data in order to obtain correct physical information on the multiplicity distributions.

## 1. Introduction

The study of multiplicity distributions (MD's) has revealed an important empirical regularity, as they are well described by a negative binomial distribution (NBD), in all reactions[1] (from  $e^+e^-$  annihilation to nucleus-nucleus collisions), in full phase space (at the highest energies, after disentangling events with different jet topology in  $e^+e^-$  annihilation[2] and different components in  $\bar{p}p$  collisions[3]) and in symmetric rapidity intervals. On one hand, the fact that the NBD fits all available data within 10% should be emphasized[4]; on the other hand, deviations from the fits should be carefully studied. One possible deviation lies in the exact shape of the tail of the distribution, which can be studied in general by analyzing (unnormalized) factorial moments of the distribution  $P_n$ :

$$F_q = \sum_{n=q}^{\infty} n(n-1)\dots(n-q+1)P_n, \quad (1)$$

or (unnormalized) factorial cumulant moments:

$$K_q = F_q - \sum_{i=1}^{q-1} \binom{q-1}{i-1} K_{q-i} F_i. \quad (2)$$

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It was recently proposed to study their ratio[5]:

$$H_q = K_q/F_q. \quad (3)$$

which is indeed very sensitive to the properties of the tail of the distribution. The theoretical basis for this analysis lies in recent results[5] obtained in perturbative QCD, within the Modified Leading Log Approximation (MLLA), which suggest that the ratio  $H_q$  of eq. (3) oscillates in sign when regarded as a function of the rank  $q$ . On the contrary, in the case of the NBD, the ratio  $H_q$  is always positive and monotonically decreasing (see eq. (10) below). It was pointed out in ref. [5] that many problems appear when one attempts to match the above mentioned QCD prediction with experimental findings; the most prominent ones are the inadequacy of MLLA calculations of moments of rank higher than the second one (higher order correlations are not under control in perturbative QCD), and the lack of knowledge of hadronization effects. Comparisons with experimental data have nonetheless appeared in the recent past[6], showing qualitative agreement with the theoretical work. This analysis however suffers from the experimental point of view of one more problem that does not apply to theory [7]: finite statistics does not allow a precise knowledge of the shape of the high multiplicity tail, because one obtains a truncated MD. This fact makes the straightforward experimental analysis questionable, as it will be shown in detail in the following section.

## 2. Truncating the multiplicity distribution

Let us consider a truncated multiplicity distribution  $P_n$ , which is zero for  $n > n_0$ ,  $n_0$  being the cutoff parameter. It is clear from the definition, eq. (1), that  $F_q = 0$  for  $q > n_0$ . It is also clear from the definition, eq. (2), that  $K_q$  is also in this case different from zero for all  $q$ . One can prove the following theorem about distributions and infinitely divisible distributions (the latter are, in our view, very important in multiparticle dynamics[8]):

*Theorem 1:* a truncated multiplicity distribution is not infinitely divisible.

The generating function of a discrete infinitely divisible distribution can always be cast in the exponential form

$$\begin{aligned} f(z) &= e^{\lambda[g(z)-1]} \\ &= e^{-\lambda} \sum_{N=0}^{\infty} \frac{1}{N!} [\lambda g(z)]^N = e^{-\lambda} \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} \left[ \sum_{m=0}^{\infty} q_m z^m \right]^N \end{aligned} \quad (4)$$

where  $\lambda$  is a constant, positive parameter and  $g(z)$  the generating function for the probabilities  $q_m$ , so that all coefficients in the summation are non-negative. Because  $g(z)$  is a generating function, one can always find  $m_0$  such that  $q_{m_0} > 0$ . Then the coefficient of  $z^{Nm_0}$  (to which  $P_{Nm_0}$  is proportional) is always larger than or equal to  $e^{-\lambda} \lambda^N q_{m_0}^N / N!$  for every  $N > 0$ , therefore the distribution  $P_n$  cannot be truncated. Hence a MD which is truncated cannot be infinitely divisible.

A second theorem can be proven on factorial cumulant moments' properties in a truncated MD:

*Theorem 2:* the factorial cumulant moments of a truncated MD are not positive definite. The combinants of a MD can be defined as follows[9]:

$$C_q = \frac{P_q}{P_0} - \frac{1}{q} \sum_{i=1}^{q-1} (q-i) C_{q-i} \frac{P_i}{P_0} \quad (5)$$

The combinants of all ranks of a MD are all non-negative if and only if the distribution is infinitely divisible[10], and since it was shown in Theorem 1 that this is not the case for a truncated distribution, it follows that at least one of the  $C_q$  is negative. But given the relation between factorial cumulant moments and combinants[9]:

$$K_q = \sum_{j=q}^{\infty} j(j-1)\dots(j-q+1) C_j \quad (6)$$

this implies that the  $K_q$  are not positive definite. Notice that since for a truncated MD  $K_q$  can be negative and  $F_q$  cannot, their ratio  $H_q$  can also change sign.

In order to visually compare the effects of truncating the multiplicity distribution on the  $H_q$ , one can take an exact MD, truncate it and compute the resulting moments. For the phenomenological reasons mentioned in the introduction, we used the NBD, whose form and moments are well known in terms of the two parameters  $\bar{n}$  and  $k$  ( $\bar{n}$  is the average number of particles and  $k$  is linked to the dispersion  $D$  by  $k^{-1} + \bar{n}^{-1} = D^2/\bar{n}^2$ ):

$$P_n^{\text{NBD}} = \frac{k(k+1)\dots(k+n-1)}{n!} \left( \frac{\bar{n}}{\bar{n}+k} \right)^n \left( \frac{k}{\bar{n}+k} \right)^k \quad (7)$$

$$F_q = \bar{n}^q \frac{(k+1)(k+2)\dots(k+q-1)}{k^{q-1}} \quad (8)$$

$$K_q = \bar{n}^q \frac{(q-1)!}{k^{q-1}} \quad (9)$$

$$H_q = \frac{(q-1)!}{(k+1)(k+2)\dots(k+q-1)} \quad (10)$$

Experimental data on charged particles MD in full phase space only present even multiplicities because of charge conservation (“even-odd effect”); therefore the truncated form that will be discussed is

$$P_n = \begin{cases} AP_n^{\text{NBD}} & \text{if } (n \text{ even}) \text{ and } (n \leq n_0) \\ 0 & \text{otherwise} \end{cases} \quad (11)$$

Here  $n_0$  is the maximum observed multiplicity and  $A$  is the normalization parameter, so that  $\sum_n P_n = 1$ ; the parameters  $\bar{n}$  and  $k$  of the complete NBD have been chosen to reproduce various experimental distributions.

In Figure 1,  $H_q$  is plotted as a function of  $q$  for the selection of values of the parameters of the NBD appearing in eq. (11) listed in Table 1: the values used were taken in order to reproduce the MD’s in  $e^+e^-$  annihilation experiments at different energies, and  $n_0$  was taken as the largest multiplicity in the published data for the same energy. Various amplitudes of oscillation are seen, depending on the parameters. Notice that the first

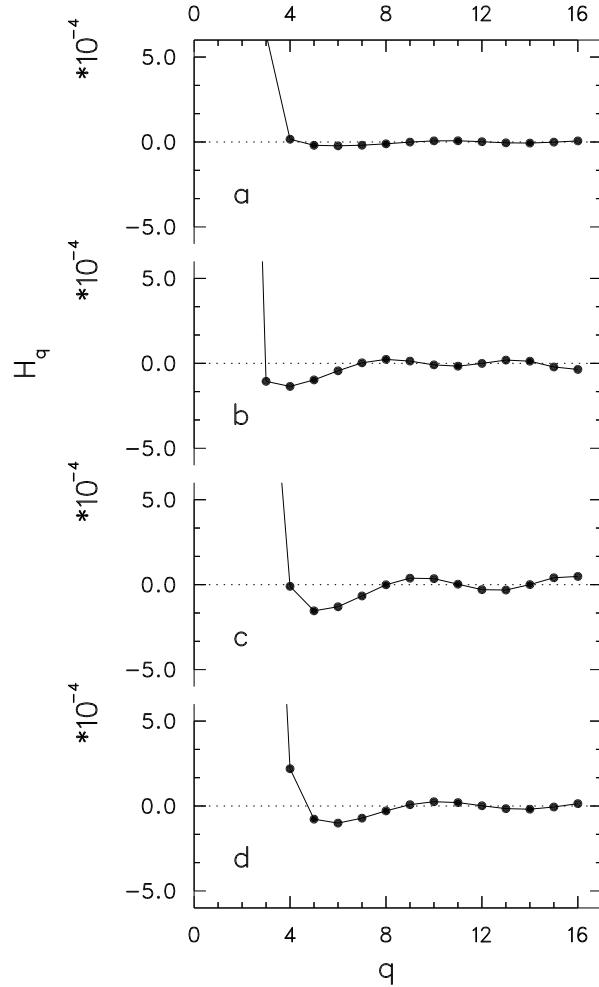


Figure 1: The ratio of factorial cumulant moments to factorial moments  $H_q$  is plotted as a function of the rank  $q$  for the distribution in eq. (11), for different choices of parameters, corresponding to a good NBD description of experimental data in  $e^+e^-$  annihilation, as listed in Table 1. The lines are drawn only to guide the eye.

Table 1: Parameters used in Figure 1. The NBD with these values of  $\bar{n}$  and  $k$  reproduces well the MD of  $e^+e^-$  annihilation data in different experiments and at different c.m. energies as listed.  $n_0$  is largest multiplicity in the published data.

	$\bar{n}$	$k$	$n_0$	Experiment
a)	12.9	212.0	28	HRS (29 GeV) [11]
b)	13.6	54.0	36	Tasso (34 GeV) [12]
c)	15.5	30.8	38	Tasso (43 GeV) [12]
d)	21.4	24.3	52	Delphi (91 GeV) [13]

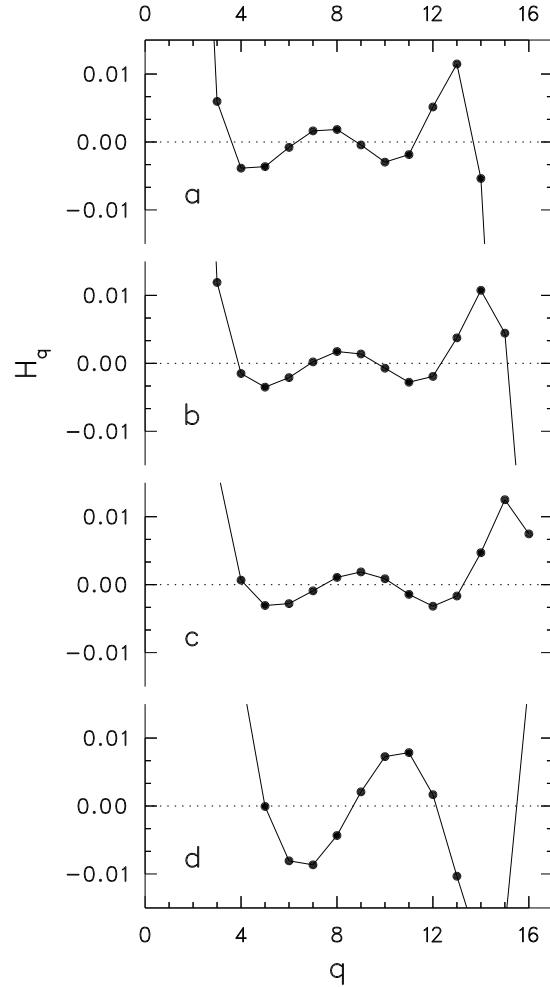


Figure 2: The ratio of factorial cumulant moments to factorial moments  $H_q$  is plotted as a function of the rank  $q$  for the distribution in eq. (11), for different choices of parameters, corresponding to a good NBD description of experimental data in  $pp$  and  $\bar{p}p$  collisions, as listed in Table 2. The lines are drawn only to guide the eye.

Table 2: Parameters used in Figure 2. The NBD with these values of  $\bar{n}$  and  $k$  reproduces well the MD of  $pp$  and  $\bar{p}p$  collisions data in different experiments and at different c.m. energies as listed.  $n_0$  is largest multiplicity in the published data.

	$\bar{n}$	$k$	$n_0$	Experiment
a)	10.7	11.0	26	ISR (30 GeV) [14]
b)	12.2	9.4	32	ISR (44 GeV) [14]
c)	13.6	8.2	38	ISR (62 GeV) [14]
d)	28.3	3.7	100	UA5 (540 GeV) [15]

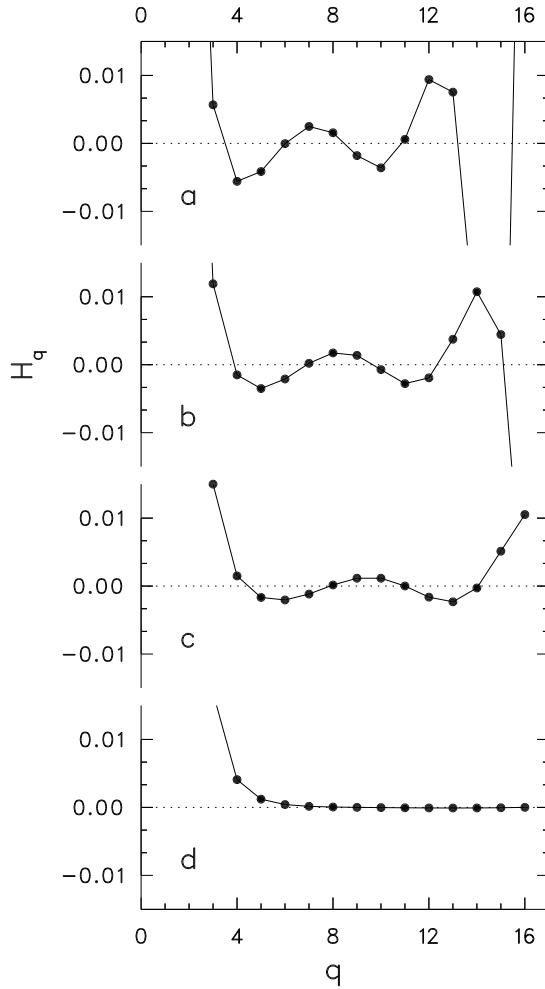


Figure 3: The ratio  $H_q$  is plotted here for  $\bar{n} = 12.2$ ,  $k = 9.4$  and a)  $n_0 = 28$ , b)  $n_0 = 32$ , c)  $n_0 = 36$ , d)  $n_0 = 64$ .

minimum lies around  $q = 5$  and that the amplitude of the oscillations are comparable to those found in the experimental data[6].

Figure 2 is similar to Figure 1, but the parameters are taken to reproduce MD's in  $pp$  and  $\bar{p}p$  collider events at different energies (see Table 2). Oscillations here are larger than in the previous case because a larger part of the MD is truncated away; their amplitudes and general shapes are still comparable to their experimental counterpart[6].

The unphysical effects of the finite statistics are clearly apparent from these figures: this fact should be taken into consideration when analyzing the ratio  $H_q$  in experimental data, and these effects should be removed from the data in order to expose physical properties.

In Figure 3 we show how the value of the cutoff  $n_0$  affects the position and amplitude of the oscillations: the other parameters are the same as in Figure 2b but  $n_0$  grows from 28 to 64 ( $n_0 = 32$  in Figure 2b as in Figure 3b.) In the last case no oscillations are visible in the figure, thus showing that such a large cutoff does not affect appreciably the ratio

$H_q$ , but lower values, comparable to the experimental maximum multiplicity, give larger oscillations because they correspond to cutting away a larger part of the distribution.

### 3. Conclusions

We have shown that the consequences of the natural fact that the experimental multiplicity distribution is truncated because of finite statistics have a large importance in the study of the ratio of factorial cumulant moments to factorial moments, and can mask physical results. An oscillatory behaviour of  $H_q$  versus  $q$  appears, which is comparable in size and shape to that observed in a straightforward analysis of experimental data as in ref. [6]. It is therefore suggested to take this effect into account in future analyses.

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